MATH1010 Assignment 2

Suggested Solution

1. (a)

$$\begin{aligned} f'(c) &= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \\ &= \lim_{x\to c} \frac{\frac{x^2}{x+1}-\frac{c^2}{c+1}}{x-c} \\ &= \lim_{x\to c} \frac{xc+x+c}{(x+1)(c+1)} \\ &= \frac{c(c+2)}{(c+1)^2} \end{aligned}$$

(b)

$$\begin{aligned} g'(c) &= \lim_{x\to c} \frac{g(x)-g(c)}{x-c} \\ &= \lim_{x\to c} \frac{x^{\frac{2}{3}}-c^{\frac{2}{3}}}{x-c} \\ &= \lim_{x\to c} \frac{(x^{\frac{1}{3}}-c^{\frac{1}{3}})(x^{\frac{1}{3}}+c^{\frac{1}{3}})}{(x^{\frac{1}{3}}-c^{\frac{1}{3}})(x^{\frac{2}{3}}+x^{\frac{1}{3}}c^{\frac{1}{3}}+c^{\frac{2}{3}})} \\ &= \frac{2}{3}c^{-\frac{1}{3}} \end{aligned}$$

2. (a) Since $e^x, \sqrt{x}, |x|$ are continuous on$[0, +\infty)$

$$\begin{aligned} \lim_{x\to 0} e^{\sqrt{|\sin x|}} &= e^{\lim\limits_{x\to 0}\sqrt{|\sin x|}} \\ &= e^{\sqrt{\lim\limits_{x\to 0}|\sin x|}} \\ &= e^{\sqrt{|\lim\limits_{x\to 0}\sin x|}} \\ &= 1 \end{aligned}$$

(b) Since $\ln(x), |x|$ is continuous on$(0, +\infty)$

$$\begin{aligned}\lim_{x\to\pi} \ln(1 + |\cos x|) &= \ln \lim_{x\to\pi}(1 + |\cos x|) \\ &= \ln(1 + |\lim_{x\to\pi} \cos x|) \\ &= \ln 2\end{aligned}$$

3. $f'(x) = -\sin x + x$

By Mean Value Theorem,for $x \geqslant 0$, we have

$$\sin x - \sin 0 = \sin'(c)(x - 0) \quad \text{for some} \quad c \in (0, x)$$

Hence $\sin x \leqslant x$ for $x \geqslant 0$

Thus $f'(x) = -\sin x + x \geqslant 0$ for $x \geqslant 0$ and $f(x)$ is an increasing function on $[0, +\infty)$

4. If $f(2) = 2, f'(2) = 3, g(2) = 4, g'(2) = 5,$

(a)

$$\begin{aligned}\frac{d}{dx}\Big(f(x)g(x)\Big)\Big|_{x=2} &= f(2)g'(2) + f'(2)g(2) \\ &= 22\end{aligned}$$

(b)

$$\begin{aligned}\frac{d}{dx}\Big(\frac{f(x)}{g(x)}\Big)\Big|_{x=2} &= \frac{g(2)f'(2) - f(2)g'(2)}{\Big(g(2)\Big)^2} \\ &= \frac{1}{8}\end{aligned}$$

(c)

$$\frac{d}{dx}\left(g(f(x))\right)\Big|_{x=2} = g'(f(2)) \cdot f'(2)$$
$$= 15$$

5. (a) Since $f(x)$ is continuous function on $\mathbb{R}$

   $\lim_{x\to1^+} f(x) = f(1) = \lim_{x\to1^-} f(x)$

   $1 + A = 1$ and $A = 0$

   (b)

$$f(x) = \begin{cases} x & \text{if} \quad x \geqslant 1 \\ x^2 - x + 1 & \text{if} \quad x < 1 \end{cases}$$

   $f(x)$ is differentiable on $x < 1$ as $f(x) = x^2 - x + 1$ for $x < 1$

   $f(x)$ is differentiable on $x > 1$ as $f(x) = x$ for $x > 1$

$$\lim_{x\to1^-} \frac{f(x) - f(1)}{x - 1} \qquad \lim_{x\to1^+} \frac{f(x) - f(1)}{x - 1}$$
$$= \lim_{x\to1^-} \frac{x^2 - x + 1 - 1}{x - 1} \qquad = \lim_{x\to1^+} \frac{x - 1}{x - 1}$$
$$= 1 \qquad = 1$$

6. (a) $f(x)$ is continuous for $x > 0$ and $x < 0$

$$\lim_{x\to0^-} f(x) = 0$$
$$\lim_{x\to0^+} f(x) = \lim_{y\to+\infty} e^{-y}$$
$$= 0$$

   $f(x)$ is continuous at $x = 0$ and hence is contiuous on $\mathbb{R}$

(b) i.

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{e^{-\frac{1}{x^2}}}{x}$$
$$= \lim_{y\to+\infty}\sqrt{y}e^{-y}$$
$$= 0$$
$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = 0$$

Therefore, $f(x)$ is differentiable at 0

ii.

$$f'(x) = \begin{cases} \frac{2e^{-\frac{1}{x^2}}}{x^3} & x > o \\ 0 & x \leqslant 0 \end{cases}$$

(c)

$$\lim_{x\to 0^-}\frac{f'(x)-f'(0)}{x-0}$$
$$= 0$$

$$\lim_{x\to 0^+}\frac{f'(x)-f'(0)}{x-0}$$
$$= \lim_{y\to+\infty}\frac{2e^{-y}}{\frac{1}{y}}$$
$$= 0$$

$f'(x)$ is differentiable at 0.

7. (a) Suppose $n = 0$,

$$f(x) = \begin{cases} \cos^2\frac{1}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

We claim $\lim\limits_{x\to 0} f(x) \neq 0$:
Let $\varepsilon = \frac{1}{2}$, for any $\delta > 0$, there exist $k_0 \in \mathbb{N}$ such that

$$0 < \frac{1}{2k_0\pi} < \delta \quad \text{and} \quad f(\frac{1}{2k_0\pi}) = 1 > \frac{1}{2}$$

$f(x)$ is not continuous at $x = 0$

(b) i. Suppose $n = 1$,

$$f(x) = \begin{cases} x\cos^2\frac{1}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

$$-1 \leqslant \cos^2\frac{1}{x} \leqslant 1$$

$$-|x| \leqslant x\cos^2\frac{1}{x} \leqslant |x| \quad \text{for} \quad x \neq 0$$

By Sandwich Theorem,$\lim_{x\to 0} x\cos^2\dfrac{1}{x} = 0$

ii.

$$\begin{aligned} \lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0^+}\frac{x\cos^2\frac{1}{x}}{x} \\ &= \lim_{x\to 0^+}\cos^2\frac{1}{x} \qquad \text{does no exist} \end{aligned}$$

$f(x)$ is not differentiable on $\mathbb{R}$

iii. $f'(x) = \cos^2\frac{1}{x} + \frac{1}{x}\sin\frac{2}{x}$ for $x \neq 0$

iv. Since $f'(0)$ is not defined, f' is not continuous on $\mathbb{R}$

(c) Suppose $n = 2$,

i. Since $\lim_{x\to 0} x^2\cos^2\dfrac{1}{x} = 0$, f is continuous at $x = 0$

ii. Since $\lim_{x\to 0}\dfrac{x^2\cos\frac{1}{x}}{x} = 0$, f is differentiable on $\mathbb{R}$

iii.

$$f'(x) = \begin{cases} 2x\cos^2\frac{1}{x} + 2\cos\frac{1}{x}\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

iv. $\lim_{x\to 0} 2x\cos^2\dfrac{1}{x} + 2\cos\dfrac{1}{x}\sin\dfrac{1}{x}$ doex not exist
So f' is not continuous on $\mathbb{R}$

8. (a) True

   Assume $f(x)+g(x)$ is continuous at $x=c$

   Since $f(x)$ is continuous at $x=c$

   Then $g(x)=\Big(f(x)+g(x)\Big)-f(x)$ is continuous at $x=c$

   (Contradiction arise)

   Therefore, $f(x)+g(x)$ is not continuous at $x=c$.

   (b) False

   Consider $f(x)=\begin{cases} 1 & x \geqslant 0 \\ -1 & x<0 \end{cases}$, $g(x)=\begin{cases} -1 & x \geqslant 0 \\ 1 & x<0 \end{cases}$

   Then $f(x)+g(x)=0$ for $x \in \mathbb{R}$ (which is continuous)

   (c) False

   Consider $f(x)=|x|$ and $g(x)=\begin{cases} 1 & \text{if} \quad x>0 \\ -1 & \text{if} \quad x \leqslant 0 \end{cases}$

   Then $f(x)$ is continuous at $x=0$, $g(x)$ is not continuous at $x=0$,

   but $f(x)g(x)=x$ is continuous at $x=0$

   (d) False

   Let $f(x)=0$ for all $x \in \mathbb{R}$

   $g(x)=\begin{cases} 1 & x=0 \\ 0 & x \neq 0 \end{cases}$

   g is not continuous at $x=f(1)$

   f is continuous at $x=1$

   But $g \circ f$ is continuous at $x=1$

9. (a) Let $f(x)=\sqrt{|x|}$ is continuous functions on $\mathbb{R}$

   By the Theorem,$f(g(x))=\sqrt{|g(x)|}$ is also a continuous function on $\mathbb{R}$

(b) Let $f(x) = \begin{cases} x & \text{if} \quad x \geqslant 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$ is a continuous function on $\mathbb{R}$ By the Theorem,$h(x) = f(g(x))$ is a continuous function on $\mathbb{R}$